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Quantitative Results in the Theory of Overconvergence of Complex Interpolating Polynomials

V. TOTIK

Bolyai Institute, Szeged, Aradi V. tere 1, 6720 Hungary

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We generalize and make exact several well-known estimates concerning the overconvergence of complex interpolating polynomials. © 1986 Academic Press, Inc.

1. INTRODUCTION

Let $\rho \geqslant 1$ and denote by A_ρ and $A_\rho C$ the set of all functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k(f) z^k$$

that are analytic in the circle $|z| < \rho$ and have singularity on $|z| = \rho$, and analytic in $|z| < \rho$ and continuous on $|z| = \rho$, respectively. We set

$$P_{n-1,j}(f; z) = \sum_{k=0}^{n-1} a_{k+jn} z^k, \qquad j = 0, 1, \ldots,$$

$$Q_{n-1,l}(f; z) = \sum_{k=0}^{l-1} P_{n-1,j}(f; z), \qquad l = 1, 2, \ldots,$$

and denote by $L_{n-1}(f; z)$ the Lagrange interpolating polynomial of f of degree at most $(n-1)$ based on the nth roots of unity. Finally, we put

$$\Delta_{l,n-1}(f; z) = L_{n-1}(f; z) - Q_{n-1,l}(f; z).$$

Generalizing a result of Walsh [4, p. 153], Cavaretta, Sharma and Varga [1] proved

THEOREM A. *For any $f \in A_\rho$, $\rho > 1$, and for any positive integer l we have, for $R \geqslant \rho$,*

$$f_l(R) \stackrel{\text{def}}{=} \varlimsup_{n \to \infty} \max_{|z| = R} |\Delta_{l,n-1}(f; z)|^{1/n} \leqslant R/\rho^{l+1}. \tag{1}$$

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In particular, $\Delta_{l,n-1}(f;z)$ converges to zero as $n\to\infty$ for every $|z|<\rho^{l+1}$ (this is where the term "overconvergence" comes from). Our first result is that in (1) actually the equality holds.

Let

$$\kappa_l(R,\rho)=R/\rho^{l+1} \qquad \text{if} \quad R\geqslant\rho$$
$$=1/\rho^l \qquad \text{if} \quad 0\leqslant R<\rho.$$

THEOREM 1. *If* $f\in A_\rho$, $\rho>1$, l *is a positive integer and* $R>0$ *then* $f_l(R)=\kappa_l(R,\rho)$.

COROLLARY 1. *If* $l\geqslant 1$, f *is analytic in an open domain containing* $|z|\leqslant 1$ *and* $f_l(R)=\kappa_l(R,\rho)$ *for some* $R>0$, $\rho>1$ *then* $f\in A_\rho$.

For example, if we know a priori that f is holomorphic on $|z|\leqslant 1$ and if

$$L_{n-1}(f;z)-Q_{n-1,l}(f;z)$$

is uniformly bounded in every closed subdomain of $|z|<\rho^{l+1}$ then f is analytic in $|z|<\rho$. An interesting result of Szabados [3] states that this is true if we know merely $f\in A_1C$ (cf. also the Remark in [3]).

Problem. Is Theorem 1 true for $\rho=1$ if we assume $f\in A_1C$?

Remark. For $R=0$ Theorem 1 is no longer true. Indeed (cf. below)

$$\Delta_{l,n-1}(f;0)=\sum_{j=l}^{\infty}a_{ln}+\mathcal{O}((\rho-\varepsilon)^{-(l+1)}), \qquad (\varepsilon>0)$$

and it may happen that every $a_{ln}=0$.

After this let us focus our attention on the behaviour of $\Delta_{l,n-1}(f;z)$ on $|z|=\rho^{l+1}$. Let

$$\Delta_{l,n-1}(f)=\max_{|z|=\rho^{l+1}}|\Delta_{l,n-1}(f;z)|$$
$$=\max_{|z|=\rho^{l+1}}|L_{n-1}(f;z)-Q_{n-1,l}(f;z)|.$$

By Theorem 1

$$\overline{\lim_{n\to\infty}}\,\Delta_{l,n-1}^{1/n}(f)=1$$

but this estimate is too rough; it does not tell anything about the convergence of $\Delta_{l,n-1}(f)$ to zero. A finer result is the following in which $\phi(n)\sim\phi(2n)$ means $1/c\leqslant\phi(2n)/\phi(n)\leqslant c$, $n=1,2,\ldots$, for some positive c.

THEOREM 2. *Let $f \in A_\rho$, $\rho > 1$, $l \geqslant 1$ and $\{\phi(n)\}$ a positive monotonic sequence with $\phi(2n) \sim \phi(n)$. Then the two statements*

$$\text{(i)}\ \Delta_{l,n-1}(f) = \mathcal{O}(\phi(n))$$

and

$$\text{(ii)} \qquad a_n(f) = \mathcal{O}(\rho^{-n}\phi(n))$$

are equivalent.

Note that (ii) *is independent of l, therefore* (i) *holds or not simultaneously for all $l \geqslant 1$.*

COROLLARY 2. *If $f \in A_\rho$, $\rho > 1$, $l \geqslant 1$ then*

$$\lim_{n \to \infty} \Delta_{l,n-1}(f; z) = 0 \tag{2}$$

uniformly on $|z| = \rho^{l+1}$ if and only if $a_n(f) = o(\rho^{-n})$ $(n \to \infty)$.

This solves the following problem of Szabados ([3, Problem 2]) in the negative: Assume $\rho > 1$, $l \geqslant 1$, $f \in A_1 C$ *and* (2). Does this imply $f \in A_\rho C$? By Corollary 1 any function $f \in A_\rho \setminus A_\rho C$ with a_n $(f) = o(\rho^{-n})$ testifies the negative answer.

COROLLARY 3. *If $f \in A_\rho C$, $\rho > 1$, $l \geqslant 1$, and $\Delta_{l,n-1}(f) = \mathcal{O}(n^{-1})$ then the Taylor expansion of f converges uniformly on $|z| = \rho$.*

COROLLARY 4. *If $f \in A_\rho C$, $\rho > 1$, $l \geqslant 1$, $\alpha > 1$ and $\Delta_{l,n-1}(f) = \mathcal{O}(n^{-\alpha})$ then the Taylor expansion of f converges absolutely on $|z| = \rho$.*

Remarks 1. Using the above-mentioned result of Szabados it follows that Corollaries 2 and 4 hold also with the assumption $f \in A_1 C$ instead of $f \in A_\rho$.

2. The proof of Theorem 2 shows that (i) and (ii) are also equivalent to the following: for fixed $0 < R \neq \rho$

$$\Delta_{l,n-1}(f; z) = \mathcal{O}((\kappa_l(R, \rho))^n \, \phi(n))$$

uniformly on $|z| = \rho$.

3. $f \in A_\rho C$ and $\Delta_{l,n-1}(f) = \mathcal{O}(n^{-\alpha})$, $\alpha < 1$ do not imply the uniform convergence of the Taylor expansion of f on $|z| = \rho$ and $\Delta_{l,n-1}(f) = \mathcal{O}(n^{-1})$ does not imply its absolute convergence on $|z| = \rho$ (cf. Corollaries 3 and 4).

Indeed, using Theorem 2 and the change of variable $z'=z/\rho$ we have to show that if $f\in A_1C$ and $a_n(f)=\mathcal{O}(n^{-\alpha})$ $(0<\alpha<1)$ or $a_n(f)=\mathcal{O}(n^{-1})$ then

$$\sum_{n=0}^{\infty} a_n(f)z^n$$

need not converge uniformly or absolutely on $|z|=1$, respectively. Putting

$$S_{n,m,r}(z)=\sum_{k=r}^{m}\frac{1}{k}(z^{n+k}-z^{n-k}) \qquad (r\leqslant m\leqslant n)$$

(these are essentially the well-known Fejér polynomials) we have $|S_{n,m,r}(z)|\leqslant 10$ $(|z|=1)$, and so the function f defined by

$$f(z)=\sum_{n=2}^{\infty} n^{-2}S_{4^n,4^n-1,[4^{(n-1)\alpha}]}(z) \qquad (0<\alpha<1)$$

proves the first statement while

$$f(z)=\sum_{k=1}^{\infty} a_k S_{2n_k,n_k,[n_k a_k]}(z)$$

proves the second one where $a_k\geqslant 0$, $\sum_k a_k<\infty$, $\sum_k a_k\log(1/a_k)=\infty$ and $\{n_k\}$ increases sufficiently rapidly.

Our next concern will be the pointwise behaviour of $\Delta_{l,n-1}(f;z)$. Saff and Varga [2] recently proved

THEOREM B. *If $f\in A_\rho$, $\rho>1$ and $l\geqslant 1$ then the sequence $\{\Delta_{l,n-1}(f;z)\}_{n=1}^{\infty}$ can be bounded in at most l distinct points in $|z|>\rho^{l+1}$.*

A more exact result is the following one.

THEOREM 3. *Let $f\in A_\rho$, $\rho>1$ and $l\geqslant 1$. Then*

(i) $$\overline{\lim_{n\to\infty}}\, |\Delta_{l,n-1}(f;z)|^{1/n}=|z|/\rho^{l+1}$$

for all but at most l points in $|z|>\rho$,

(ii) $$\overline{\lim_{n\to\infty}}\, |\Delta_{l,n-1}(f;z)|^{1/n}=1/\rho^{l}$$

for all but at most $(l-1)$ points in $0<|z|<\rho$.

(iii) *if $\{\phi(n)\}_{n=1}^{\infty}$ is monotone, $\phi(2n)\sim\phi(n)$ and*

$$\Delta_{l,n-1}(f;z_j)=\mathcal{O}(\phi(n)) \qquad (j=1,\dots,l+1)$$

in some $(l+1)$ *points* $z_1,..., z_{l+1}$ *with* $|z_1| = \cdots = |z_{l+1}| = \rho^{l+1}$ *then* $a_n(f) = \mathcal{O}(\rho^{-n}\phi(n))$ *and hence*

$$\Delta_{l,n-1}(f; z) = \mathcal{O}(\phi(n))$$

uniformly on $|z| = \rho^{l+1}$.

Note that (i), (ii) and (iii) are a certain strengthening of one half of Theorem 1 and Theorem 2, respectively.

COROLLARY 5. *If* $f \in A_\rho$, $\rho > 1$ *and* $l \geqslant 1$ *then there are only two possibilities*:

(i) $\lim_{n\to\infty} \Delta_{l,n-1}(f; z) = 0$ *uniformly on* $|z| = \rho^{l+1}$, *and*

(ii) $\lim_{n\to\infty} \Delta_{l,n-1}(f; z) = 0$ *in at most* l *points on* $|z| = \rho^{l+1}$.

Furthermore, by Corollary 2, either (i) or (ii) holds simultaneously for all $l \geqslant 1$.

In connection with Theorem B, Shaff and Varga [2] also proved that its statement is the best possible one in a certain sense. Now we show that Theorem 3 cannot be improved.

THEOREM 4. *Let* $\rho > 1$ *and* $l \geqslant 1$.

(i) *If* $z_1,..., z_l$ *are arbitrary* l *points with modulus greater than* ρ *then there is a rational function* $f \in A_\rho$ *with*

$$\overline{\lim_{n\to\infty}} |\Delta_{l,n-1}(f; z_j)|^{1/n} < \frac{|z_j|}{\rho^{l+1}}, \qquad j = 1, 2,..., l. \tag{3}$$

(ii) *If* $z_1,..., z_{l-1}$ *are arbitrary* $(l-1)$ *points in the ring* $0 < |z| < \rho$ *then there is a rational function* $f \in A_\rho$ *with*

$$\overline{\lim_{n\to\infty}} |\Delta_{l,n-1}(f; z_j)|^{1/n} < \frac{1}{\rho^l}, \qquad j = 1,..., l-1.$$

Our proof will show that if $\rho^{l+1} \leqslant |z_j| < \rho^{l+2}$ then the function f in (i) can be chosen to satisfy

$$\Delta_{l,n-1}(f; z_j) = o(1), \qquad j = 1,..., l.$$

This is the mentioned result of Shaff and Varga.

In his pioneering article Walsh also verified that his overconvergence result (Theorem A)) cannot be extended to $|z| > \rho^2$. Indeed, for $f(z) = 1/(\rho - z)$, $\{\Delta_{1,n-1}(f; z)\}$ does not tend to zero if $|z| = \rho^2$. This special result may be considered as the appearance of the more general

THEOREM 5. *If* $f \in A_\rho$, $\rho > 1$, $l \geqslant 1$ *and* f *has a pole on* $|z| = \rho$ *then* $\{\Delta_{l,n-1}(f; z)\}_{n=1}^{\infty}$ *can tend to zero in at most* l *points on* $|z| = \rho^{l+1}$.

By Theorem 4 this is the best possible result. We obtain also that for functions $f \in A_\rho$ having a pole on $|z| = \rho$ always the second alternative holds in Corollary 5.

2. PROOFS

Proof of Theorem 1. Since $f \in A_\rho$ if and only if

$$\overline{\lim_{k \to \infty}} \, |a_k|^{1/k} = 1/\rho \tag{4}$$

we have $a_k = \mathcal{O}((\rho - \varepsilon)^{-k})$ for every $\varepsilon > 0$. Let R be fixed, $|z| = R$ and if $R < \rho$ then we assume $\varepsilon > 0$ so small that $R < \rho - \varepsilon$ be statisfied, as well. Then we obtain by a formula of Szabados [3] the estimate

$$\begin{aligned} \Delta_{l,n-1}(f; z) &= \sum_{k=0}^{n-1} \sum_{j=l}^{\infty} a_{k+jn} z^k \\ &= \sum_{k=0}^{n-1} a_{ln+k} z^k + \mathcal{O}\left(\sum_{k=0}^{n-1} |z|^k (\rho - \varepsilon)^{(l-1)n-k} \right) \\ &= \sum_{k=0}^{n-1} a_{ln+k} z^k + \mathcal{O}\begin{cases} (R/(\rho-\varepsilon)^{l+2})^n & \text{if} \quad R \geqslant \rho \\ (1/(\rho-\varepsilon)^{l+1})^n & \text{if} \quad 0 < R < \rho. \end{cases} \end{aligned} \tag{5}$$

Whence

$$\begin{aligned} |\Delta_{l,n-1}(f; z)| &\leqslant K \sum_{k=0}^{n-1} R^k (\rho - \varepsilon)^{ln+k} + \mathcal{O}\begin{cases} (R/(\rho-\varepsilon)^{l+2})^n \\ (1/(\rho-\varepsilon)^{l+1})^n \end{cases} \\ &\leqslant K \begin{cases} (R/(\rho-\varepsilon)^{l+1})^n & \text{if} \quad R \geqslant \rho \\ (1/(\rho-\varepsilon)^{l})^n & \text{if} \quad 0 < R < \rho \end{cases} \end{aligned}$$

by which $f_l(R) \leqslant \kappa_l(R, \rho - \varepsilon)$. Since here $\varepsilon > 0$ was arbitrary, we obtain $f_l(R) \leqslant \kappa_l(R, \rho)$.

To prove the opposite inequality let first $R \geqslant \rho$, and let $\varepsilon > 0$ be so small that

$$(\rho - \varepsilon)^{-(l+2)} < \rho^{(l+1)} \tag{6}$$

is satisfied. If $m = ln + k$, where $n - l - 1 \leqslant k \leqslant n - 1$; then by (5)

$$\begin{aligned} |a_m| &= \left| \frac{1}{2\pi i} \int_{|z|=R} \frac{\Delta_{l,n-1}(f; z)}{z^{k+1}} \, dz \right| + \mathcal{O}\left(\frac{1}{R^{k+1}} \left(\frac{R}{(\rho-\varepsilon)^{l+2}} \right)^n \right) \\ &\leqslant K (f_l(R) + \varepsilon)^n R^{-k} + ((\rho - \varepsilon)^{-(l+2)}) \end{aligned}$$

and seeing that $k \sim n$, $n(l+1) \sim m$ we obtain from (4) and (6) that

$$f_l(R) + \varepsilon \geqslant R \varlimsup_{n \to \infty} \{|a_m| - \mathcal{O}((\rho - \varepsilon)^{-(l+2)n})\}^{1/n}$$

$$= R\left(\varlimsup_{n \to \infty} |a_m|^{1/m}\right)^{m/n} = R/\rho^{l+1},$$

which proves

$$f_l(R) \geqslant \kappa_l(R, \rho).$$

For $0 < R < \rho$ we obtain similarly from (5) that for $0 \leqslant k < l$, $m = ln + k$

$$|a_m| \leqslant KR^{-k}(f_l(R) + \varepsilon)^n + \mathcal{O}((\rho - \varepsilon)^{-(l+1)n} R^{-k})$$

by which

$$f_l(R) \geqslant \varlimsup_{n \to \infty} |a_m|^{1/n} = 1/\rho^l$$

and the proof is complete.

Corollary 1 immediately follows from Theorem 1 because $\kappa_l(R, \rho) \neq \kappa_l(R, \rho')$ if $\rho \neq \rho'$.

Proof of Theorem 2. If $\{\phi(n)\}$ is monotone and $\phi(2n) \sim \phi(n)$ then there is a constant c with $(1/c)\, n^{-c} < \phi(n) < cn^c$. Hence, following the consideration of the preceding proof we obtain that $a_n(f) = \mathcal{O}(\rho^{-n}\phi(n))$ implies

$$\Delta_{l,n-1}(f) \leqslant K\phi(n) \sum_{k=0}^{n-1} \rho^{-(ln+k)} \rho^{(l+1)k} + o(\phi(n)) \leqslant K\phi(n)$$

and conversely, $\Delta_{l,n-1}(f) = \mathcal{O}(\phi(n))$ implies

$$|a_m| \leqslant K\rho^{-k(l+1)}\phi(n) + o(\phi(n)\, \rho^{-(l+1)n}) \leqslant K\rho^{-m}\phi(m)$$

$$(m = ln + k,\ n - l - 1 \leqslant k \leqslant n - 1)$$

and the proof is complete.

If $a_n(f) = o(\rho^{-n})$ then there is a sequence $\{\phi(n)\}$, $\phi(2n) \sim \phi(n)$ monotonically decreasing to 0 such that $|a_n(f)| \leqslant K\rho^{-n}\phi(n)$. By Theorem 2, this implies $\Delta_{l,n-1}(f) = o(1)$ $(n \to \infty)$, and the first part of Corollary 2 is proved. The necessity of $a_n(f) = o(\rho^{-n})$ can be similarly proved.

Corollary 4 directly follows from Theorem 2.

If we assume $\Delta_{l,n-1}(f)=\mathcal{O}(n^{-1})$, then we have, by Theorem 2, $a_n(f)=\mathcal{O}(n^{-1})$. On the other hand, $f\in A_\rho C$ implies that the $(C,1)$-means of the Taylor series of f converge uniformly on $|z|=\rho$ to f. Thus, Corollary 3 follows from the Tauberian theorem of Hardy (see [5, p. 78]).

Proof of Theorem 3. Let first $|z|>\rho$. By (5) we have for sufficiently small $\varepsilon>0$

$$\begin{aligned}
h(z) &=^{\text{def}} \Delta_{l,n-1}(f;z)-z^l\Delta_{l,n}(f;z)\\
&=\sum_{k=0}^{l-1} a_{ln+k}z^k-\sum_{k=0}^{l} a_{(l+1)n+k)}z^{n+k}\\
&\quad+\mathcal{O}((|z|(\rho-\varepsilon)^{-(l+2)})^n)\\
&=-\sum_{k=0}^{l} a_{(l+1)n+k}z^{n+k}+\mathcal{O}((\rho-\varepsilon)^{-ln}+(|z|(\rho-\varepsilon)^{-(l+2)})^n)\\
&=-\sum_{k=0}^{l} a_{(l+1)n+k}z^{n+k}+\mathcal{O}\left(\left(\frac{|z|}{\rho^{l+1}}-\eta\right)^n\right)
\end{aligned}$$

where η is a positive number.

If we assume

$$\overline{\lim_{n\to\infty}}\ |\Delta_{l,n-1}(f;z_j)|^{1/n}<|z_j|/\rho^{l+1},\qquad j=1,\dots,l+1$$

for $z_1,\dots,z_{l+1}$ with $|z_1|,\dots,|z_{l+1}|>\rho$ then we have also

$$\overline{\lim_{n\to\infty}}\ |h(z_j)|^{1/n}<|z_j|/\rho^{l+1},\qquad j=1,\dots,l+1$$

and so, by the above estimate on h, there are number $\eta_1>0$, $K_1\geqslant 1$ and $\beta_{j,n}$, $n=1,2,\dots$, $1\leqslant j\leqslant l+1$, such that

$$|\beta_{j,n}|<K_1\left(\frac{|z_j|}{\rho^{l+1}}-\eta_1\right)^n$$

and

$$\sum_{k=0}^{l} a_{(l+1)n+k}z_j^k=z_j^{-n}\beta_{j,n},\qquad 1\leqslant j\leqslant l+1.$$

Solving this system of equations for $a_{(l+1)n+k}$ we obtain

$$a_{(l+1)n+k} = \sum_{j=1}^{l+1} c_j^{(k)} z_j^{-n} \beta_{j,n}$$

with appropriate constants $c_j^{(k)}$ independent of n, by which

$$\begin{aligned}\overline{\lim_{n\to\infty}} \ & |a_{(l+1)n+k}|^{1/((l+1)n+k)} \\ & \leqslant K_1 \left(\overline{\lim_{n\to\infty}} \left(\frac{1}{\rho^{l+1}} - \frac{\eta_1}{\max|z_j|} \right)^{n/((l+1)n+k)} \right) \\ & < 1/\rho\end{aligned}$$

independently of $0 \leqslant k \leqslant l$, which contradicts (4). This contradiction proves statement (i).

In the proof of (ii), one can argue similarly using the estimate

$$\begin{aligned} h(z) &= \sum_{k=0}^{l-1} a_{ln+k} z^k + \mathcal{O}((|z|(\rho-\varepsilon)^{-(l+1)})^n + (\rho-\varepsilon)^{-(l+1)n}) \\ &= \sum_{k=0}^{l-1} a_{ln+k} z^k + \mathcal{O}\left(\left(\frac{1}{\rho^l} - \eta\right)^n\right). \end{aligned}$$

The proof of (iii) is almost the same as that of (i) (see also the proof of Theorem 2).

Finally, Corollary 5 follows from assertion (iii) exactly as Corollary 2 does from Theorem 2.

Proof of Theorem 4. Let us consider the system of equations

$$\sum_{k=0}^{l} a_{(l+1)m+k} z_j^k = 0, \qquad j = 1, 2, ..., l \tag{7}$$

where $a_{(l+1)m+k}$ are the unknowns and $m = 0, 1, ...$. Solving this for $a_{(l+1)m+1}, ..., a_{(l+1)m+l}$ we obtain that there are numbers c_k (independent of m) with

$$a_{(l+1)m+k} = c_k a_{(l+1)m}, \qquad m = 1, 2, ..., k = 1, ..., l.$$

Let $c_0 = 1$ and

$$f(z) = \left(\sum_{k=0}^{l} c_k z^k \right) \Bigg/ \left(1 - \left(\frac{z}{\rho} \right)^{l+1} \right).$$

Then f is a rational function and $f \in A_\rho$ (f has at least one pole on $|z| = \rho$).

Writing the denominator of f in the form

$$\sum_{m=0}^{\infty}\left(\frac{z}{\rho}\right)^{(l+1)m}$$

we obtain that

$$a_{(l+1)m+k}(f)=\rho^{-(l+1)m}c_k$$

and thus these numbers $a_{(l+1)m+k}=a_{(l+1)m+k}(f)$ satisfy (7). For any $n>0$ let r and s be determined by $ln+s=(l+1)r$, $0\leqslant s<l+1$. Using (7) we obtain for $n>0$.

$$\begin{aligned}\sum_{k=0}^{n-1} a_{ln+k}z_j^k &= \sum_{k=0}^{s-1} a_{ln+k}z_j^k+\sum_{m=r}^{n-1} z_j^{(l+1)m-ln}\sum_{k=0}^{l} a_{(l+1)m+k}z_j^k\\ &=\sum_{k=0}^{s-1} a_{ln+k}z_j^k=\mathcal{O}(\rho^{-ln}).\end{aligned}$$

This and (5) yield for every $\varepsilon>0$

$$\Delta_{l,n-1}(f;z_j)=\mathcal{O}(\rho^{-ln}+(|z_j|/(\rho-\varepsilon)^{l+2})^n), \qquad j=1,2,..., l$$

and putting here an $\epsilon>0$ for which (6) is satisfied we get the desired relation (3).

The proof of (ii) is similar, only one has to solve the system of equations $c_0=1$

$$\sum_{k=0}^{l-1} c_k z_j^k=0, \qquad j=1,..., l-1$$

for $c_0,..., c_{l-1}$ and then put

$$f(z)=\left(\sum_{k=0}^{l-1} c_k z^k\right)\Big/\left(1-\left(\frac{z}{\rho}\right)^l\right).$$

The proof is complete.

Proof of Theorem 5. Let z_0, $|z_0|=\rho$, be a pole of f. This means that f can be extended to a neighborhood of z_0 such that the extended function has a pole at z_0. Then

$$\lim_{\substack{z\to z_0\\ |z|<\rho}} |f(z)|\,|z-z_0|>0. \tag{8}$$

Now if the conclusion of the theorem does not hold then, by Corollary 5 and Theorem 2, $a_n(f)=o(\rho^{-n})$ $(n\to\infty)$, and so

$$\begin{aligned}\lim_{r\to 1-0}|f(rz_0)|\,|rz_0-z_0| &= |z_0| \lim_{r\to 1-0}\left(\sum_{n=0}^{\infty} o((r\rho/\rho)^n)\right)(1-r)\\ &= o\left(\lim_{r\to 1-0}(1/(1-r))(1-r)\right)\\ &= o(1)\end{aligned}$$

contradicting (8).

Our proofs are complete.

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